

# Hurwitz Integrality of Power Series Expansion of The Sigma Function for a Plane Curve

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## Abstract

This paper shows Hurwitz integrality of the coefficients of expansion at the origin of the sigma function  $\sigma(u)$  associated to a certain plane curve which should be called a plane telescopic curve. For the prime 2, the expansion of  $\sigma(u)$  is not Hurwitz integral, but  $\sigma(u)^2$  is. This paper clarifies the precise structure of this phenomenon. Throughout the paper, computational examples for the trigonal genus three curve ((3, 4)-curve)  $y^3 + (\mu_1 x + \mu_4)y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8)y = x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}$  ( $\mu_j$  are constants) are given.

## Introduction

1. More than twenty years ago, Buchstaber, Enolskii, and Leykin started to investigate certain family of plane algebraic curves, which they call  $(n, s)$ -curves. Around that time, Shinji Miura also wrote some papers for the purpose of coding theory on larger class of curves includes many non-plane curves and the family above. His papers are published only in Japanese. The paper [11] is one of his most important papers. An important subclass of his class is called the class of telescopic curves, in which any curve has many good properties in a contrast to the other curves. Especially, they have good number theoretic properties as I described below. For example, like the coefficient of the elliptic curves, any coefficient of a telescopic curve has weight, and good generalization of the sigma function of Weierstrass can be attached. In this paper, we only treat *plane* curves introduced by Buchstaber, Enolskii, and Leykin, and show that the power series expansion of the attached sigma function is Hurwitz integral (see the definition 1.7) outside the prime 2. Moreover, we show full detail of how the prime 2 appears in the expansion.

Our result includes the case for the Weierstrass sigma function. Investigations for this case are given in papers by Bannai-Kobayashi [2], Mazur-Stein-Tate [10], Mazur-Tate [9] from other points of view. Our result in this paper might be generalized to any telescopic curves.

2. For any Abelian variety  $A$  with a choice of a theta divisor which does not pass through the origin of  $A$ , some  $p$ -adic theta functions are constructed in [4] and [3]. In such case, the power series expansion of those theta series around the origin are Hurwitz integral. However, for any telescopic curve, the standard theta divisor passes through the origin of the Jacobian variety. Since, in general, the theta divisor has some singularity at the origin, the author was not able to prove Hurwitz integrality by a similar method to [4] and [3]. In fact, the expansion of the sigma function for a telescopic curve is not completely Hurwitz integral and has some defect at the prime 2. So, the author believe that to publish this paper is worthwhile.

3. The sigma function for a non-singular algebraic curve of genus  $g$  is an entire function of  $g$  variables which is invariant under any change (modular transform) of the symplectic base of the homology group of the curve and whose zeroes are of order 1 and exactly along the pull-back image of the  $(g - 1)$ st symmetric product of copies of the curves with respect to the map given by modulo the lattice. Because of its invariance under the modular transform, the expansion of the sigma function around the origin is expressed only in terms of the coefficients of the defining equation of the curve. In fact, the coefficients of the expansion are polynomials over the rationals of the coefficients of the defining equation of the curve. Actually, the expansion is Hurwitz integral over the ring generated by the coefficients of the defining equation over the *integers* outside the prime 2, and is treated, as it is without modifications, not only over the complex numbers but also over  $p$ -adic numbers or other rings.

4. In Section 1, we introduce basic situation and notations. In Section 6, following Nakayashiki [12], we show an expression of the sigma function as a determinant of infinite size (the tau function) times an exponential function. In Section 3, we prove the integrality

of an expansion of the canonical Klein's fundamental 2-form. In Section 7, after analyzing the exponential part, we complete the proof. As multivariate sigma functions are not well-known, we present fairly detailed proofs as well as many examples.

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**6.** We use the usual notation  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{C}$  for the ring of integers, the field of rationals, the field of complex numbers. Finally, we note here that the 1-forms of the first kind are included our definition of 1-forms of the second kind, in this paper.

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# 1 Statement of the main theorem

Let  $e$  and  $q$  be fixed two positive integers such that  $e < q$  and  $\gcd(e, q) = 1$ . Let us define, for these integers, a polynomial of  $x$  and  $y$

$$(1.1) \quad f(x, y) = y^e + p_1(x)y^{e-1} + \cdots + p_{e-1}(x)y - p_e(x),$$

where  $p_j(x)$  is a polynomial of  $x$  of degree  $\lceil \frac{jq}{e} \rceil$  or below and its coefficients are denoted as

$$(1.2) \quad \begin{aligned} p_j(x) &= \sum_{k: jq-ek > 0} \mu_{jq-ek} x^k \quad (1 \leq j \leq e-1), \\ p_e(x) &= x^q + \mu_{e(q-1)} x^{q-1} + \cdots + \mu_{eq}. \end{aligned}$$

The base ring over which we set up situation is quite general. However, for simplicity we start by letting it an algebraically closed field and assuming  $\mu_i$  to be constants belonging to the field. Let  $\mathcal{C}$  be the projective curve defined by

$$(1.3) \quad f(x, y) = 0$$

having unique point  $\infty$  at infinity. This should be called an  $(e, q)$ -curve following to Buchstaber, Enolskii, and Leykin, or *plane Miura curve* after the paper [11]. If this is non-singular, the genus of it is  $(e-1)(q-1)/2$ . Whether  $\mathcal{C}$  is non-singular or singular, we constantly denote this by  $g$ :

$$g = \frac{(e-1)(q-1)}{2}.$$

As general elliptic curve is defined by an equation of the form

$$(1.4) \quad y^2 + (\mu_1 x + \mu_3) y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6,$$

our curve  $\mathcal{C}$  is a natural generalization of elliptic curves.

We introduce weight as follows:

$$\text{wt}(\mu_j) = -j, \quad \text{wt}(x) = -e, \quad \text{wt}(y) = -q.$$

Then all the equations for functions, power series, differential forms, and so on in this paper are of homogeneous weight. Needless to say that  $\text{wt}(f(x, y)) = -eq$ .

Here, we shall explain what is the sigma function roughly. Let  $J = \text{Jac}(\mathcal{C})$  be the Jacobian variety of  $\mathcal{C}$ . Now, we take the *standard theta divisor*  $\Theta^{[g-1]} \subset J$  that is strictly defined in (4.2) below. Then the *sigma function* attached to  $\mathcal{C}$  is a meromorphic section of the sheaf  $\mathcal{O}(\Theta^{[g-1]})$ . In other words, it is an entire function defined over the universal (Abelian) covering  $\mathbb{C}^g$  of  $J$  which is invariant under the modular transformation associated to the natural symplectic structure and the second derivatives of logarithm of it are periodic function with respect to the lattice determined by the chosen symplectic base of  $H_1(\mathcal{C}, \mathbb{Z})$ . So that, the sigma function has zeroes of order 1 along the pull-back divisor of  $\Theta^{[g-1]} \subset \text{Jac}(\mathcal{C})$  under  $\mathbb{C}^g \rightarrow J$  and invariant under the transformation by the natural

action of  $\mathrm{Sp}(2g, \mathbb{Z})$ . Such function is uniquely determined up to overall multiplication of a constant. The modular invariance implies that the power series expansion of the sigma function is expressed only by data on  $\mathcal{C}$  such as its coefficients. In fact, the power series expansion of the sigma function around the origin has polynomial coefficients in  $\mu_j$ 's over  $\mathbb{Q}$  ([13]).

In order to define (see 5) the sigma function classically, we take the point  $\infty$  to be the base point of  $\mathcal{C}$ , and fix a symplectic base of  $H_1(\mathcal{C}, \mathbb{Z})$  and of  $H^1(\mathcal{C}, \mathbb{C})$  as we explain below. As usual, the base of later one consist of  $g$  differential forms of the first kind and  $g$  ones of the second kinds having a pole only at  $\infty$ .

Assuming that we have defined the sigma function strictly, we shall state the main result after the following two more definitions.

**Definition 1.5.** For each coefficient  $\mu_j$  of  $f(x, y)$ , we denote  $\mu_j' = \frac{1}{2}\mu_j$  if  $\mu_j$  is the coefficient of monomial with odd power of  $x$  times odd power of  $y$ , and  $\mu_j' = \mu_j$  otherwise. Moreover, we denote by  $\boldsymbol{\mu}'$  the set of all  $\mu_j'$ .

**Example 1.6.** If  $(e, q) = (3, 4)$ , then

$$f(x, y) = y^3 + (\mu_1 x + \mu_4) y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8) y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}).$$

Hence,  $\boldsymbol{\mu}' = \{\mu_1, \mu_4, \mu_8, \mu_2, \frac{1}{2}\mu_5, \mu_3, \mu_6, \mu_9, \mu_{12}\}$ .

**Definition 1.7.** Let  $R$  be an integral domain with characteristic 0 and  $z_1, z_2, \dots, z_n$  be indeterminates. Then we define

$$R\langle\langle z_1, z_2, \dots, z_n \rangle\rangle = \left\{ \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} a_{j_1 j_2 \cdots j_n} \frac{z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}}{j_1! j_2! \cdots j_n!} \mid a_{j_1 j_2 \cdots j_n} \in R \right\}.$$

This is a commutative ring by usual addition and multiplication. If a power series with coefficients in the quotient field of  $R$  belongs to this ring, it is said to be *Hurwitz integral* over  $R$ .

The sigma function  $\sigma(u)$  is a function of  $g$  variables. The entries of the variable  $u$  are suffixed by the sequence of positive integers  $\{w_g, w_{g-1}, \dots, w_1\}$  that are not expressed as  $ae + bq$  with positive integers  $a$  and  $b$ , which are called the *Weierstrass gaps* for  $(e, q)$ , and is denoted as  $u = (u_{w_g}, u_{w_{g-1}}, \dots, u_{w_1})$ . The main result of this paper is the following.

**Theorem 1.8.** *Using the notation above, we have the following:*

*The power series expansion of the sigma function  $\sigma(u) = \sigma(u_{w_g}, u_{w_{g-1}}, \dots, u_{w_1})$  attached to the curve  $\mathcal{C}$  defined by  $f(x, y) = 0$  of (1.1) has the following properties:*

$$\begin{aligned} \sigma(u) &\in \mathbb{Z}[\boldsymbol{\mu}']\langle\langle u_{w_g}, \dots, u_{w_1} \rangle\rangle, \\ \sigma(u)^2 &\in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle u_{w_g}, \dots, u_{w_1} \rangle\rangle. \end{aligned}$$

**Remark 1.9.** (1) To the best knowledge of the author, there are no example such that  $\sigma(u) \in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle u_{w_g}, \dots, u_{w_1} \rangle\rangle$ .

(2) Let  $x^i y^j$  be a term of  $f(x, y)$  with both of  $i$  and  $j$  being odd. The suffix  $dq - di - qj$  of its coefficient  $\mu_{dq-di-qj}$  belongs to the Weierstrass gap sequence (2.5) for the pair  $(d, q)$ .

Let us give an illustration.

**Example 1.10.** We quote first several terms of  $\sigma(u)$  for  $(e, q) = (3, 4)$  from Theorem 7.1 in [6] Dividing the terms by weight with respect to  $(u_{w_g}, \dots, u_{w_1})$  only as

$$\sigma(u) = C_5 + C_6 + C_7 + C_8 + C_9 + \dots,$$

then we see

$$\begin{aligned} C_5 &= u_5 - u_1 u_2^2 + 6 \frac{u_1^5}{5!}, \\ C_6 &= 2\mu_1 \frac{u_1^4 u_2}{4! 1!} - 2\mu_1 \frac{u_2^3}{3!}, \\ C_7 &= 10(\mu_1^2 - 3\mu_2) \frac{u_1^7}{7!} + 2\mu_2 \frac{u_1^3 u_2^2}{3! 2!}, \\ C_8 &= 2(\mu_1^3 + 9\mu_3 - 2\mu_1 \mu_2) \frac{u_1^6 u_2}{6! 1!} - 6\mu_3 \frac{u_1^2 u_2^3}{2! 3!}, \\ C_9 &= 14(\mu_1^2 - 3\mu_2)^2 \frac{u_1^9}{9!} + 2(2\mu_4 - \mu_2^2 + \mu_1^2 \mu_2 + 6\mu_1 \mu_3) \frac{u_1^5 u_2^2}{5! 2!} \\ &\quad - 2(4\mu_1 \mu_3 + 4\mu_4 + \mu_2^2) \frac{u_1 u_2^4}{1! 4!} + 2\mu_4 \frac{u_1^4 u_5}{4! 1!}, \\ &\quad \dots\dots\dots \end{aligned}$$

Observing these terms, they looks like Hurwitz integral.

Our main result 1.8 claims that the expansion is Hurwitz integral over  $\mathbb{Z}[\mu_1, \mu_4, \mu_2, \frac{1}{2}\mu_5, \mu_8, \mu_3, \mu_6, \mu_9, \mu_{12}]$  (we need to divide only  $\mu_5$  by 2). According to a computation by J.C. Eilbeck, we have

$$\begin{aligned} &\frac{3}{2}\mu_5^2 \frac{u_1^5 u_5^2}{5!2!} \quad [15], \quad -\frac{1}{2}\mu_5^2 \frac{u_1 u_2^2 u_5^2}{2!2!} \quad [15], \quad \frac{1}{4}\mu_5^2 \frac{u_5^3}{3!} \quad [15], \\ &-\frac{15}{2}\mu_2 \mu_5^2 \frac{u_1^7 u_5^2}{7!2!} \quad [17], \quad \frac{1}{2}\mu_2 \mu_5^2 \frac{u_1^3 u_2^2 u_5^2}{3!2!2!} \quad [17], \\ &\frac{63}{2}\mu_2^2 \mu_5^2 \frac{u_1^9 u_5^2}{9!2!} \quad [19], \quad -\frac{1}{2}u_2^2 \mu_2^2 \mu_5^2 \frac{u_1^5 u_2^2 u_5^2}{5!2!2!} \quad [19], \quad -\frac{1}{2}\mu_2^2 \mu_5^2 \frac{u_1 u_2^4 u_5^2}{2!4!} \quad [19], \\ &\frac{1}{4}\mu_5^3 \frac{u_1^3 u_2 u_5^3}{3!3!} \quad [20]. \end{aligned}$$

Here, each number in [ ] indicates the weight of the term with respect to  $u_{w_j}$ s.

This paper is organized as follows. Using the result in [12] of Nakayashiki, the sigma function is expressed as a product of the following two functions. The first one is a determinant of infinite size whose entries are coefficients of functions with only pole at  $\infty$  with respect to certain local parameter  $t$  at  $\infty$ , and are indexed by  $\mathbb{N} \times \mathbb{N}$ , where  $\mathbb{N}$  is the set of positive integers. The other function is an exponential of power series defined by using some coefficients of expansion with respect to  $t$  of certain differential 1-form of the first kind, and of expansion of so-called Klein's differential 2-form for  $\mathcal{C}$ , which is determined from the  $2g$  forms representing a base of  $H^1(\mathcal{C}, \mathbb{C})$ . The former one has zeroes

on the same place with  $\sigma(u)$  and is expanded at the origin as a power series of Hurwitz integral over  $\mathbb{Z}[\boldsymbol{\mu}]$ . However, the second derivatives of this determinant are not periodic functions. By multiplying the later exponential function, we get periodic functions. Such exponential function is expanded as a power series not of Hurwitz integral at the prime 2 (over  $\mathbb{Z}[\boldsymbol{\mu}]$ ). So, our proof of the main theorem is divided into two parts corresponding to such two factors of  $\sigma(u)$ .

## 2 Arithmetic local parameter

In this section, we define a special local parameter called the *arithmetic local parameter*. Let  $e$  and  $q$  be two coprime positive integers such that  $q > e$ . Consider the pairs  $(a, b)$  of integers such that

$$ae - bq = 1.$$

We chose the pair  $(a, b)$  such that the absolute value  $|a|$  is minimal in the such pairs. Note that, in this case,  $|b|$  is also minimal. We define  $\text{sign}(a)$  to be 1 or  $-1$  according to the sign of  $a$ . Let

$$c = -2a + \text{sign}(a)q, \quad d = -2b + \text{sign}(a)e.$$

Then we see

$$(2.1) \quad -ce + dq = 2.$$

Especially, we have

$$|a| < \lceil q/2 \rceil, \quad |b| < \lceil e/2 \rceil, \quad |c| < \lceil q/2 \rceil, \quad |d| < \lceil e/2 \rceil,$$

and that

$$\begin{aligned} (ad - bc)e &= d + 2b, & |d + 2b| &< \frac{3}{2}(e + 1); \\ (ad - bc)q &= c + 2a, & |c + 2a| &< \frac{3}{2}(q + 1). \end{aligned}$$

Because of (2.1), we have

$$ad - bc = \text{sign}(a).$$

Therefore, the four integers  $a, b, c$ , and  $d$  are simultaneously positive or negative. Now for the curve  $\mathcal{C}$  defined by (1.3), we let

$$t = x^{-a}y^b, \quad s = x^c y^{-d}.$$

Then  $t$  is a local parameter at  $\infty$ . The parameter  $t$  has nice properties as we will see. We call this  $t$  the *arithmetic local parameter* of  $\mathcal{C}$ . Obviously, the weights are  $\text{wt}(t) = 1$ ,  $\text{wt}(s) = 2$ , and

$$(2.2) \quad x = t^{-|d|} s^{-|b|}, \quad y = t^{-|c|} s^{-|a|}.$$

Plugging this into (1.3), we let

$$\begin{aligned} (2.3) \quad \tilde{f}(t, s) &= f\left(\frac{1}{t^{|d|} s^{|b|}}, \frac{1}{t^{|c|} s^{|a|}}\right) t^{|c|e} s^{|a|e} \cdot \begin{cases} t^2 & (a, b, c, d > 0) \\ -s & (a, b, c, d < 0) \end{cases} \\ &= -t^2 + s + \cdots \in \mathbb{Z}[\boldsymbol{\mu}][t, s]. \end{aligned}$$

In particular,  $\text{wt}(\tilde{f}(t, s)) = 2$ .

If  $x$  and  $y$  satisfy  $f(x, y) = 0$ , then the corresponding  $t$  and  $s$  satisfy  $\tilde{f}(t, s) = 0$ . Then, using (2.3) recursively,  $s$  is expressed as a power series of  $t$  such that

$$s = t^2 + \cdots \in t^2\mathbb{Z}[\boldsymbol{\mu}][[t]].$$

Moreover,  $x = x(t)$  and  $y = y(t)$  are expanded as power series of  $t$ ;

$$\begin{aligned} x(t) &= t^{-e} + \cdots \in t^{-e}\mathbb{Z}[\boldsymbol{\mu}][[t]], \\ y(t) &= t^{-q} + \cdots \in t^{-q}\mathbb{Z}[\boldsymbol{\mu}][[t]]. \end{aligned}$$

**Example 2.4.** For the case  $(e, q) = (3, 4)$ , we have the following :

$$\begin{aligned} f(x, y) &= y^3 + (\mu_1 x + \mu_4) y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8) y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}), \\ \tilde{f}(t, s) &= -t^2 + s + (-\mu_3 t^3 + \mu_1 t) s + (-\mu_6 t^4 + \mu_5 t^3 + \mu_4 t^2) s^2 + (-\mu_9 t^5 + \mu_8 t^4) s^3 - \mu_{12} t^6 s^4, \\ x(t) &= t^{-3} + \mu_1 t^{-2} - \mu_3 + \mu_4 t + (-\mu_4 \mu_1 + \mu_5) t^2 + (\mu_4 \mu_1^2 - \mu_5 \mu_1 - \mu_6) t^3 + \cdots, \\ y(t) &= t^{-4} + \mu_1 t^{-3} - \mu_3 t^{-1} + \mu_4 + (-\mu_4 \mu_1 + \mu_5) t + (\mu_4 \mu_1^2 - \mu_5 \mu_1 - \mu_6) t^2 + \cdots. \end{aligned}$$

We are going to express our differential forms given by  $(x, y)$ -coordinates in terms of  $(s, t)$ . We display in increasing order the integers belonging to  $\{me + nq \mid m, n = 0, 1, 2, 3, \dots\}$  as follows:

$$0 = m_0 e + n_0 q, \quad e = m_1 e + n_1 q, \quad \dots, \quad m_j e + n_j q, \quad \dots$$

It is well-known that  $1, 2g - 1$ , and all the integers greater than or equal to  $2g$  appear in this sequence, and exactly  $g$  integers less than  $2g$  appear. Let

$$(2.5) \quad w_1 (= 1), \quad w_2, \quad \dots, \quad w_g (= 2g - 1)$$

be the subsequence in increasing order of the integers less than  $2g$ . These are exactly the Weierstrass gaps at the point  $\infty$  of  $\infty \in \mathcal{C}$ . For  $j = 0, \dots, g - 1$ , we define

$$(2.6) \quad \omega_{w_{g-j}}(t) = \omega_{w_{g-j}}(x, y) = -\frac{x^{m_j} y^{n_j} dx}{f_y(x, y)} = (t^{w_{g-j}-1} + \cdots) dt \in t^{w_{g-j}-1} \mathbb{Z}[\boldsymbol{\mu}][[t]] dt.$$

Then,  $\text{wt}(\omega_{w_{g-j}}(t)) = w_{g-j}$  (we set  $dt = 1$ ), and the set of these forms makes a base of the space of the differential forms of the first kind.

In order to express  $\omega_{w_j}$  in terms of  $t$  and  $s$  algebraically, we use notation

$$\begin{Bmatrix} w \\ v \end{Bmatrix} = \begin{cases} w & (a, b, c, d > 0) \\ v & (a, b, c, d < 0) \end{cases}.$$

From

$$f(x, y) = \tilde{f}(t, s) t^{-|c|e} s^{-|a|e} \cdot \begin{Bmatrix} -t^{-2} \\ s^{-1} \end{Bmatrix},$$

we see that

$$f_y(x, y) = \tilde{f}_s(t, s) t^{-|c|e+|c|+|d|+1} s^{-|a|e+|a|+|b|+1} \frac{dx}{dt} \cdot \begin{Bmatrix} t^{-2} \\ s^{-1} \end{Bmatrix}.$$



Therefore, we have

$$(2.7) \quad \frac{dx}{f_y(x, y)} = \frac{t^{|c|e-|c|-|d|-1} s^{|a|e-|a|-|b|-1}}{\tilde{f}_s(t, s)} \cdot \left\{ \begin{matrix} t^2 \\ s \end{matrix} \right\} dt.$$

**Example 2.8.** In the case  $(e, q) = (3, 4)$ , we have

$$\begin{aligned} \omega_5(x, y) &= -\frac{dx}{f_y} = \frac{st^2 dt}{\tilde{f}_s} \\ &= (t^4 - 2\mu_1 t^5 + (3\mu_1^2 - 2\mu_2)t^6 + (-4\mu_1^3 + 6\mu_2\mu_1 + 2\mu_3)t^7 + \cdots) dt, \\ \omega_2(x, y) &= -\frac{xdx}{f_y} = \frac{tdt}{\tilde{f}_s} \\ &= (t - \mu_1 t^2 + \mu_1^2 t^3 + (-\mu_1^3 + \mu_3)t^4 + \cdots) dt, \\ \omega_1(x, y) &= -\frac{ydx}{f_y} = \frac{dt}{\tilde{f}_s} \\ &= (1 - \mu_1 t + (\mu_1^2 - \mu_2)t^2 + (-\mu_1^3 + 2\mu_2\mu_1 + \mu_3)t^3 + \cdots) dt. \end{aligned}$$

### 3 Klein's fundamental 2-form

In this section we define *Klein's fundamental 2-form*, and investigate its Hurwitz integrality. Klein's fundamental 2-form is useful for computing a natural symplectic base of the vector space

$$(3.1) \quad H^1(\mathcal{C}, \mathbb{C}) \simeq \varinjlim_n H^0(\mathcal{C}, d\mathcal{O}(n\infty)) / \varinjlim_n dH^0(\mathcal{C}, \mathcal{O}(n\infty))$$

(see [13] or [14]).

**Definition 3.2.** Take two variable points  $(x, y)$  and  $(z, w)$  on  $\mathcal{C}$ . For the arithmetic local parameter  $t$ , let  $t_1$  and  $t_2$  be its values at these points, and define

$$(3.3) \quad [t_1, t_2]dt_1 = -\omega_{w_g}(t_1) \frac{1}{x-z} \frac{f(Z, y) - f(Z, w)}{y-w} \Big|_{Z=z},$$

where  $Z$  is an indeterminate.

**Proposition 3.4.** We use the notation (3.3). For  $i = 1, \dots, g$ , there exists a form of the second kind  $\eta_{-w_i}(t)$  of weight  $-w_i$  in  $\frac{dt}{t^{w_i+1}} \mathbb{Z}[\boldsymbol{\mu}][[t]]$  satisfying, for

$$\boldsymbol{\xi}(t_1, t_2) = \frac{d}{dt_2} [t_1, t_2] dt_1 dt_2 + \sum_{i=1}^g \omega_{w_i}(t_1) \eta_{-w_i}(t_2),$$

the following two equations:

$$(3.5) \quad \boldsymbol{\xi}(t_1, t_2) \in \frac{dt_1 dt_2}{(t_2 - t_1)^2} + \mathbb{Z}[\boldsymbol{\xi}][[t_1, t_2]] dt_1 dt_2,$$

$$(3.6) \quad \boldsymbol{\xi}(t_1, t_2) = \boldsymbol{\xi}(t_2, t_1).$$

$\boldsymbol{\xi}(t_1, t_2)$  is homogeneous of weight 2. Those  $\eta_{-w_i}(t)$  are not unique.

We call  $\xi(t_1, t_2)$  above *Klein's fundamental 2-form*.

**Remark 3.7.** Nakayashiki [13] constructed Klein's fundamental 2-form analytically by using prime forms which are expressed in terms of first derivatives of a theta function, following [7]. However, in view of our aim, we shall construct it algebraically.

*Proof.* We fix an primitive  $e$ -th root of 1 and denote it by  $1^{1/e}$ . Take the pull-back images of the function  $t \mapsto x(t)$ , and denote them by  $t, t^{(1)}, t^{(2)}, \dots, t^{(e-1)}$ . Precisely speaking, we assume that each  $t^{(j)}$  has value such that  $x(t) = x(t^{(j)})$  for the given  $t$  and satisfies

$$t^{(j)} = 1^{1/e} t + \dots \in \mathbb{Q}[[t]].$$

Using the notation of Definition 3.2, we define

$$[t_1, t_2] = -\frac{\omega_{2g-1}(x, y)}{dt_1} \frac{1}{x-z} \frac{f(Z, y) - f(Z, w)}{y-w} \Big|_{Z=z}.$$

Then

$$\lim_{t_2 \rightarrow t_1} \frac{t_1 - t_2}{x-z} \frac{f(Z, y) - f(Z, w)}{y-w} \Big|_{Z=z} = \frac{f_y(x, y)}{\frac{d}{dt}x(t_1)}.$$

So that, we see

$$(3.8) \quad \lim_{t_2 \rightarrow t_1} (t_1 - t_2)[t_1, t_2] = 1.$$

By Weierstraß preparation theorem, we have

$$x(t_1)^{-1} - x(t_2)^{-1} = (t_1 - t_2)(t_1 - t_2^{(1)})(t_1 - t_2^{(2)}) \cdots (t_1 - t_2^{(e-1)}) p(t_1, t_2)$$

with some  $p(t_1, t_2) \in 1 + (t_1, t_2)\mathbb{Z}[\mu][[t_1, t_2]]$ . Therefore, the expansions with respect to  $t_2$  of  $t_2^{(1)}, \dots, t_2^{(e-1)}$  are all belong to  $\mathbb{Z}[[t_2]]$ . Moreover, we have

$$\begin{aligned} & \frac{1}{x-z} \frac{f(Z, y) - f(Z, w)}{y-w} \Big|_{Z=z} \\ &= \frac{-(xz)^{-1}}{x^{-1} - z^{-1}} \frac{f(Z, y) - f(Z, w)}{y-w} \Big|_{Z=z} \\ &= \frac{1}{t_1 - t_2} \cdot \frac{y(t_1) - y(t_2^{(1)})}{t_1 - t_2^{(1)}} \cdot \frac{y(t_1) - y(t_2^{(2)})}{t_1 - t_2^{(2)}} \cdots \frac{y(t_1) - y(t_2^{(e-1)})}{t_1 - t_2^{(e-1)}} \\ & \quad \cdot (-x(t_1)x(t_2))^{-1} p(t_1, p_2)^{-1}. \end{aligned}$$

Here the middle  $e-1$  factors above should be reduced by Weierstraß preparation theorem. This means that, for  $j = 1, 2, \dots, e-1$ ,

$$\frac{y(t_1) - y(t_2^{(j)})}{t_1 - t_2^{(j)}} \in \bigcup_{r=1}^q t_1^{-r} (t_2^{(j)})^{-q+r-1} \mathbb{Z}[\mu][[t_1, t_2^{(j)}}].$$

Consequently, we see

$$(3.9) \quad (t_2 - t_1) \frac{1}{x-z} \frac{f(Z, y) - f(Z, w)}{y-w} \Big|_{Z=z} \in t_1^e t_2^e \bigcup_{r=e-1}^{q(e-1)} t_1^{-r} t_2^{-(q-1)(e-1)+r} \mathbb{Z}[\mu][[t_1, t_2^{(j)}}].$$

Since

$$\omega_{2g-1}(t_1) \in t_1^{(q-1)(e-1)-1} \mathbb{Z}[\boldsymbol{\mu}][[t_1]] dt_1,$$

(3.8) and (3.9) imply that

$$(3.10) \quad \begin{aligned} & (t_2 - t_1) \frac{\omega_{2g-1}(t_1)}{dt_1} \frac{1}{x - z} \frac{f(Z, y) - f(Z, w)}{y - w} \Big|_{Z=z} \\ & \in 1 + (t_2 - t_1) \bigcup_{r=-1}^{(q-1)(e-1)+1} t_1^r t_2^{2-r} \mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2^{(i)}]]. \end{aligned}$$

The space of differential form on  $\mathcal{C}$  having unique pole only at  $\infty$  at most of order  $(w_g + 1)$  are spanned by the forms of first kinds and some other  $w_g$  forms of the second kind. Obviously, those are of the form

$$(3.11) \quad \frac{x^k y^\ell}{f_y(x, y)} dx \quad (k, \ell = 0, 1, 2, \dots).$$

From the result above,  $\frac{d}{dt_2}[t_1, t_2]dt_1dt_2$ , as a form of  $t_1$ , has pole at  $t_2$  with essential part  $1/(t_1 - t_2)^2$ , and as a form of  $t_2$ , has only poles at  $t_1$  and 0. Accordingly, if we subtract from  $\frac{d}{dt_2}[t_1, t_2]dt_1dt_2$  a 2-form which is a bi-linear combination

$$(3.12) \quad \sum_{i=1}^g \omega_{w_i}(t_1) \eta_{-w_i}(t_2)$$

of forms in (3.11) over  $\mathbb{Z}[\boldsymbol{\mu}]$  and some forms  $\eta_{-w_i}(t)$  of the second kind with a pole at  $t = 0$  of degree  $-(w_i - 1)$  and no other poles, we get a 2-form which has, as a form of  $t_1$ , a pole at  $t_2$  with its essential part  $(t_1 - t_2)^{-2}dt_1dt_2$  and no other poles, and has, as a form of  $t_2$ , the same type pole at  $t_1$  and no other poles. Moreover, its coefficients of the expansion should be belong to  $\mathbb{Z}[\boldsymbol{\mu}]$  by our argument above. Therefore, it belongs to

$$\frac{dt_1dt_2}{(t_1 - t_2)^2} + \mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]]dt_1dt_2.$$

Since the 2-form  $\frac{d}{dt_2}[t_1, t_2]dt_1dt_2$  is of homogeneous weight with respect to  $\{\mu_j\}$ ,  $t_1$ , and  $t_2$ , the 2-form (3.12) should be of homogeneous weight.

Now, we define a square matrix  $M$  of size  $w_g + w_{g-1} + \dots + w_1$  as follows. For each  $i = g, \dots, 1$ , we take  $w_i$  numbers  $j = g + w_i - 1, g + w_i - 2, \dots, g$ , and make the pairs  $\{(i, j)\}$  be the numbering of columns of  $M$ . For each  $k = g, \dots, 1$ , we take  $w_k - 2$  numbers  $\ell = -w_k - 1, -w_k, \dots, -2$ , and make the pairs  $(k, \ell)$  be the numbering of lows of  $M$ . We define

$$M = [C_{i,j;k,\ell}],$$

where  $C_{i,j;k,\ell}$  is the coefficient of  $t_1^{w_k-1} t_2^{-\ell}$  in the expansion of  $\omega_{w_i}(t_1) \frac{x^{m_j} y^{n_j}}{f_y} \frac{dx}{dt}(t_2)$ . If we take square matrices, say  $M_j$ , of size  $w_j$  diagonally from  $M$  successively  $w_j$ , the entries below of these matrices are all 0, and each  $M_j$  is a lower triangular matrix.

(The proof is continued.) □

We shall illustrate above argument by an example. It will be better the rest of proof also is explained through this example.

**Example 3.13.** If  $(e, q) = (3, 4)$ , then

$$\begin{aligned}
& [t_2, t_1] + \frac{1}{t_2 - t_1} \\
&= -\frac{1}{t_1} \\
&+ (t_1^4 - 2\mu_1 t_1^5 + \dots) \frac{1}{t_2^5} \\
&+ (2\mu_1 t_1^4 - 4\mu_1^2 t_1^5 + \dots) \frac{1}{t_2^4} \\
&+ ((\mu_1^2 + 2\mu_2)t_1^4 - (2\mu_1^3 + 4\mu_2\mu_1)t_1^5 + \dots) \frac{1}{t_2^3} \\
&+ (t_1 - \mu_1 t_1^2 + (\mu_1^2 - \mu_2)t_1^3 + (-\mu_1^3 + 4\mu_2\mu_1)t_1^4 \\
&\quad + (\mu_1^4 - 7\mu_2\mu_1^2 - 2\mu_4 + \mu_2^2)t_1^5 + \dots) \frac{1}{t_2^2} \\
&+ (1 - \mu_2 t_1^2 + (\mu_2\mu_1 + \mu_3)t_1^3 + (-\mu_2\mu_1^2 - 2\mu_3\mu_1 + 2\mu_2^2)t_1^4 \\
&\quad + (\mu_2\mu_1^3 + 3\mu_3\mu_1^2 - 4\mu_2^2\mu_1 - 2\mu_3\mu_2 - 2\mu_5)t_1^5 + \dots) \frac{1}{t_2} \\
&+ (\mu_1 + (-\mu_1^2 + \mu_2)t_1 + (\mu_1^3 - 2\mu_2\mu_1)t_1^2 + \dots) \\
&+ (-\mu_4 t_1^2 + (2\mu_4\mu_1 - \mu_5)t_1^3 + (-3\mu_4\mu_1^2 + 2\mu_5\mu_1 + 2\mu_2\mu_4)t_1^4 + \dots)t_2 \\
&+ \dots.
\end{aligned}$$

Differentiating this with respect to  $t_2$ , we have

$$\begin{aligned}
& \frac{d}{dt_2}[t_1, t_2] - \frac{1}{(t_2 - t_1)^2} \\
&= \left(-5t_1^4 + 10\mu_1 t_1^5 + \dots\right) \frac{1}{t_2^6} \\
&+ \left(-8\mu_1 t_1^4 + 16\mu_1^2 t_1^5 + \dots\right) \frac{1}{t_2^5} \\
&+ \left(-(3\mu_1^2 + 6\mu_2)t_1^4 + (6\mu_1^3 + 12\mu_2\mu_1)t_1^5 + \dots\right) \frac{1}{t_2^4} \\
&+ \left(-2t_1^1 + 2\mu_1 t_1^2 + (-2\mu_1^2 + 2\mu_2)t_1^3 + (2\mu_1^3 - 8\mu_2\mu_1)t_1^4 \right. \\
&\quad \left. + (-2\mu_1^4 + 14\mu_2\mu_1^2 - 2\mu_2^2 + 4\mu_4)t_1^5 + \dots\right) \frac{1}{t_2^3} \\
&+ \left(-t_1^0 + (\mu_2)t_1^2 - (\mu_2\mu_1 + \mu_3)t_1^3 + (\mu_2\mu_1^2 + 2\mu_3\mu_1 - 2\mu_2^2)t_1^4 \right. \\
&\quad \left. + (-\mu_2\mu_1^3 - 3\mu_3\mu_1^2 + 4\mu_2^2\mu_1 + 2\mu_3\mu_2 + 2\mu_5)t_1^5 + \dots\right) \frac{1}{t_2^2} \\
&+ \left(-\mu_4 t_1^2 + (2\mu_4\mu_1 - \mu_5)t_1^3 + (-3\mu_4\mu_1^2 + 2\mu_5\mu_1 + 2\mu_4\mu_2)t_1^4 \right. \\
&\quad \left. + (4\mu_4\mu_1^3 - 3\mu_5\mu_1^2 - 6\mu_4\mu_2\mu_1 + 2\mu_5\mu_2 - 2\mu_4\mu_3)t_1^5 + \dots\right) \\
&+ \dots.
\end{aligned} \tag{3.14}$$

The forms in a base of the space of forms of the second kind and their expansions with respect to the arithmetic parameter  $t$  with  $(x, y) = (x(t), y(t))$  are as follows:

$$\begin{aligned}
(3.15) \quad & \frac{x^2 y}{f_y(x, y)} dx = \left( \frac{-1}{t^6} - \frac{\mu_1}{t^5} - \frac{\mu_2}{t^4} + \frac{\mu_3}{t^3} + (-\mu_4^2 + \mu_8)t^2 + \dots \right) dt \in \frac{1}{t^6} \mathbb{Z}[\boldsymbol{\mu}][[t]] dt, \\
& \frac{x^3}{f_y(x, y)} dx = \left( \frac{-1}{t^5} - \frac{\mu_1}{t^4} - \frac{\mu_2}{t^3} + \frac{\mu_3}{t^2} + (-\mu_4^2 + \mu_8)t^3 + \dots \right) dt \in \frac{1}{t^5} \mathbb{Z}[\boldsymbol{\mu}][[t]] dt, \\
& \frac{y^2}{f_y(x, y)} dx = \left( \frac{-1}{t^4} + \mu_4 + (-2\mu_4\mu_1 + \mu_5)t + \dots \right) dt \in \frac{1}{t^4} \mathbb{Z}[\boldsymbol{\mu}][[t]] dt, \\
& \frac{xy}{f_y(x, y)} dx = \left( -\frac{1}{t^3} + \mu_4 t + (-2\mu_4\mu_1 + \mu_5)t^2 + \dots \right) dt \in \frac{1}{t^3} \mathbb{Z}[\boldsymbol{\mu}][[t]] dt, \\
& \frac{x^2}{f_y(x, y)} dx = \left( -\frac{1}{t^2} + \mu_4 t^2 + \dots \right) dt \in \frac{1}{t^2} \mathbb{Z}[\boldsymbol{\mu}][[t]] dt.
\end{aligned}$$

We want, by choosing suitable  $\{a_j, b_j, c_j\}$  such that

$$\begin{aligned}
(3.16) \quad & \eta_{-5}(t) = \frac{a_0 x^2 y + a_1 x^3 + a_2 y^2 + a_3 x y + a_4 x^2}{f_y(x, y)} dx, \\
& \eta_{-2}(t) = \frac{b_0 y^2 + b_1 x y}{f_y(x, y)} dx, \\
& \eta_{-1}(t) = \frac{c_0 x y}{f_y(x, y)} dx
\end{aligned}$$

satisfy (3.5). Here each of  $a_j, b_j$ , and  $c_j$  is an element in  $\mathbb{Z}[\boldsymbol{\mu}]$  of weight  $-j$  or equals to 0. These elements are obtained as a set of solutions of some linear equation whose coefficient matrix is  $M$  in the proof. The matrix  $M$  is given by the following table.

expansion in $t_1$ × expansion in $t_2$	$\frac{1}{f_y} \frac{dx}{dt}(t_1) \cdot$					$\frac{x}{f_y} \frac{dx}{dt}(t_1) \cdot$		$\frac{y}{f_y} \frac{dx}{dt}(t_1) \cdot$
	$\frac{x^2 y}{f_y} \frac{dx}{dt}(t_2)$	$\frac{x^3}{f_y} \frac{dx}{dt}(t_2)$	$\frac{y^2}{f_y} \frac{dx}{dt}(t_2)$	$\frac{xy}{f_y} \frac{dx}{dt}(t_2)$	$\frac{x^2}{f_y} \frac{dx}{dt}(t_2)$	$\frac{xy}{f_y} \frac{dx}{dt}(t_2)$	$\frac{x^2}{f_y} \frac{dx}{dt}(t_2)$	
coeff. of $t_1^4 t_2^{-6}$	1	0	0	0	0	0	0	0
coeff. of $t_1^4 t_2^{-5}$	$\mu_1$	1	0	0	0	0	0	0
coeff. of $t_1^4 t_2^{-4}$	$\mu_2$	$\mu_1$	1	0	0	0	0	0
coeff. of $t_1^4 t_2^{-3}$	$-\mu_3$	$\mu_2$	0	1	0	$-\mu_1^3 + 2\mu_2\mu_1 + \mu_3$	0	0
coeff. of $t_1^4 t_2^{-2}$	0	$-\mu_3$	0	0	1	0	$-\mu_1^3 + 2\mu_2\mu_1 + \mu_3$	$\begin{cases} \mu_1^4 - 3\mu_2\mu_1^2 \\ -2\mu_3\mu_1 - 2\mu_4 + \mu_2^2 \end{cases}$
coeff. of $t_1 t_2^{-3}$	0	0	0	0	0	1	0	0
coeff. of $t_1 t_2^{-2}$	0	0	0	0	0	0	1	$-\mu_1$
coeff. of $1 t_2^{-2}$	0	0	0	0	0	0	0	1

This table is given as follows. For example, we have  $\mu_2$  in the  $(4, 2)$ -entry. This is the coefficient of “ $t_1^4 t_2^{-3}$  in the expansion of  $\frac{1}{f_y} \frac{dx}{dt}(t_1)$  times  $\frac{x^3}{f_y} \frac{dx}{dt}(t_2)$ ”. This means that  $\mu_2$  is the coefficient  $-1$  of  $t_1^4$  in the expansion of  $\frac{1}{f_y} \frac{dx}{dt}(t_1)$  times the coefficient  $-\mu_2$  of  $\frac{1}{t_2^4}$  in the expansion of  $\frac{x^3}{f_y} \frac{dx}{dt}(t_2)$ .

Extracting this table as a  $8 \times 8$  matrix, we denote it  $M$ . In general  $M$  is a square matrix of size  $w_g + w_{g-1} + \dots + w_1$ .

**Lemma 3.17.** *The matrix  $M$  belong to the ring of square matrices of size  $w_g + w_{g-1} + \dots + w_1$  with entries in  $\mathbb{Z}[\boldsymbol{\mu}]$ , and it is a unit in that ring.*

*Proof.* The matrix  $M$  is diagonally concatenated by several lower triangular matrices  $\{M_j\}$  (they are indeed lower triangular because those entries are the coefficients of the expansion with respect to  $t_2$  of the corresponding 1-forms of the second kind which are 0) and the entries below the  $M_j$ s are all 0 (because those entries are the coefficients of the expansion with respect to  $t_1$  of the corresponding 1-forms of the first kind which are 0). Concluding from the argument above, we have proved the lemma.  $\square$

Here, we shall list up first several coefficients from (3.14):

$$\mathbf{q} = \begin{bmatrix} \text{the coefficient of } \frac{t_1^4}{t_2^6} \\ \text{the coefficient of } \frac{t_1^4}{t_2^5} \\ \text{the coefficient of } \frac{t_1^4}{t_2^4} \\ \text{the coefficient of } \frac{t_1^4}{t_2^3} \\ \text{the coefficient of } \frac{t_1^4}{t_2^2} \\ \text{the coefficient of } \frac{t_1}{t_2^3} \\ \text{the coefficient of } \frac{t_1}{t_2^2} \\ \text{the coefficient of } \frac{1}{t_2^2} \end{bmatrix} = \begin{bmatrix} -5 \\ -8\mu_1 \\ -(3\mu_1^2 + 6\mu_2) \\ 2(\mu_1^3 - 4\mu_2\mu_1) \\ \mu_2\mu_1^2 + 2\mu_3\mu_1 - 2\mu_2^2 \\ -2 \\ 0\mu_1 \\ -1 \end{bmatrix}$$

Since

$$\begin{aligned} \frac{d}{dt_2}[t_1, t_2]dt_1dt_2 - \frac{dt_1dt_2}{(t_2 - t_1)^2} &\in -\omega_5(t_1) \frac{a_0x^2y + a_1x^3 + a_2y^2 + a_3xy + a_4x^2}{f_y(x, y)} dx(t_2) \\ &- \omega_2(t_1) \frac{b_0xy + b_1x^2}{f_y(x, y)} dx(t_2) - \omega_1(t_1) \frac{c_0x^2}{f_y(x, y)} dx(t_2) + \mathbb{Z}[\boldsymbol{\mu}][[t_1, t_2]], \end{aligned}$$

we have

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_0 \\ b_1 \\ c_0 \end{bmatrix} = M^{-1}\mathbf{q} = \begin{bmatrix} 5 \\ 3\mu_1 \\ \mu_2 \\ \mu_2\mu_1 + 3\mu_3 \\ 2\mu_3\mu_1 + \mu_2^2 + 2\mu_4 \\ 2 \\ \mu_1 \\ 1 \end{bmatrix}.$$

So, we obtained

$$\begin{aligned} \eta_{-5}(t) &= \frac{5x^2y + 3\mu_1x^3 + \mu_2y^2 + \mu_3xy + 2\mu_1\mu_3x^2}{f_y(x, y)} dx(t) \\ &= (5t^{-6} + 6\mu_2t^{-4} - 2\mu_3t^{-3} + \mu_2^2t^{-2} + \mu_6 + \cdots)dt, \\ \eta_{-2}(t) &= \frac{2xy + \mu_1x^2}{f_y(x, y)} dx(t) \\ &= (2t^{-3} - 2\mu_5t^2 + \cdots)dt, \\ \eta_{-1}(t) &= \frac{x^2}{f_y(x, y)} dx(t) \\ &= (t^{-2} - \mu_5t^3 + \cdots)dt. \end{aligned}$$

*Proof.* (continuation.) In the next place, we shall modify the obtained

$$\xi(t_1, t_2) = \frac{d}{dt_2}[t_1, t_2]dt_1dt_2 + \sum_{i=1}^g \omega_{w_i}(t_1) \eta_{-w_i}(t_2) \in \frac{dt_1dt_2}{(t_2 - t_1)^2} + \mathbb{Z}[\xi][[t_1, t_2]]dt_1dt_2$$

in a symmetric form with respect to the exchange for  $t_1$  and  $t_2$ . For the above  $\xi$ , the 2-form on  $\mathcal{C}$

$$\xi(t_1, t_2) - \xi(t_2, t_1)$$

is holomorphic because the term  $1/(t_1 - t_2)^2$  is canceled out. Therefore, it can be written as

$$\sum_{i,j=1}^g a_{i,j} \omega_{w_i}(t_1) \omega_{w_j}(t_2),$$

where  $a_{i,j}$ s are in  $\mathbb{Z}[\xi]$  with  $a_{i,j} = -a_{j,i}$  ( $\forall i, \forall j$ ). Especially,  $a_{i,i} = 0$ . Then, since

$$\begin{aligned} & \left( \xi(t_1, t_2) - \sum_{i>j} a_{i,j} \omega_{w_i}(t_1) \omega_{w_j}(t_2) \right) - \left( \xi(t_2, t_1) - \sum_{i>j} a_{i,j} \omega_{w_i}(t_2) \omega_{w_j}(t_1) \right) \\ &= \left( \xi(t_1, t_2) - \sum_{i>j} a_{i,j} \omega_{w_i}(t_1) \omega_{w_j}(t_2) \right) - \left( \xi(t_2, t_1) - \sum_{j>i} a_{j,i} \omega_{w_j}(t_2) \omega_{w_i}(t_1) \right) \\ &= \sum_{i=1}^g a_{i,i} \omega_{w_i}(t_1) \omega_{w_i}(t_2) = 0, \end{aligned}$$

we redefine  $\xi(t_1, t_2)$  as the 2-form

$$\xi(t_1, t_2) - \sum_{i>j} a_{i,j} \omega_{w_i}(t_1) \omega_{w_j}(t_2).$$

This means if we replace  $\eta_{-w_i}(t_2)$  for  $i = 1, \dots, g$  by

$$\eta_{-w_i}(t_2) - \sum_{i>j} a_{i,j} \omega_{w_j}(t_2),$$

all the desired conditions are satisfied. □

For instance, in the  $(3, 4)$ -curve case, it is necessary to replace only  $\eta_{-5}(t)$  by

$$\begin{aligned} \eta_{-5}(t) = & (5x^2y + 3\mu_1x^3 + \mu_2y^2 + \mu_3xy + 2\mu_1\mu_3x^2 \\ & + (\mu_6 + \mu_4\mu_2)y + (\mu_1\mu_6 + \mu_2\mu_5 + \mu_3\mu_4)x) \frac{1}{f_y(x, y)} dx(t). \end{aligned}$$

The finally obtained Klein's fundamental 2-form is

$$\begin{aligned} \xi((x, y), (z, w)) &= \frac{F(x, y, z, w)dx dz}{(x - z)^2 f_y(x, y) f_y(z, w)} \\ &= \left( \frac{1}{(t_2 - t_1)^2} + \mu_4 t_1^2 + 2\mu_4 t_2 t_1 + \mu_4 t_2^2 \right. \\ &\quad \left. + (-2\mu_4 \mu_1 + \mu_5)(t_1^3 + t_2^3) + (-4\mu_4 \mu_1 + 2\mu_5)(t_2 t_1^2 + t_2^2 t_1) + \dots \right) dt_1 dt_2, \end{aligned}$$

where

$$\begin{aligned}
F(x, y, z, w) = & 3y^2w^2 - 2(xyz^3 + x^3zw) + (x^2z^2w + x^2yz^2) + (\mu_8\mu_4 + 3\mu_{12})(y + w) \\
& + \mu_8(y^2 + w^2) + (\mu_{12}\mu_1 + \mu_9\mu_4 + \mu_8\mu_5)(x + z) + (\mu_8\mu_1 + \mu_9)(xy + zw) \\
& + (2\mu_4^2 - 2\mu_8)yw + (\mu_5\mu_4 + 2\mu_9)(yz + wx) + (2\mu_4\mu_1 - \mu_5)(yxw + yzw) \\
& + (\mu_4\mu_2 + \mu_6)(yz^2 + wx^2) + 2\mu_4(y^2w + yw^2) + \mu_5(y^2z + xw^2) \\
& + 2\mu_1(xyw^2 + y^2zw) + \mu_2(y^2z^2 + x^2w^2) + (2\mu_9\mu_1 + 2\mu_8\mu_2 + 2\mu_4\mu_6 + \mu_5^2)xz \\
& + (\mu_5\mu_1 + 2\mu_6)(xzw + xyz) + (\mu_6\mu_1 + \mu_5\mu_2 + \mu_4\mu_3)(xz^2 + x^2z) \\
& + (2\mu_1^2 - 2\mu_2)xyzw + (2\mu_3\mu_1 + \mu_2^2 + 2\mu_4)x^2z^2 + (\mu_2\mu_1 + 3\mu_3)(xyz^2 + x^2zw) \\
& + \mu_1(x^2z^3 + x^3z^2) + (\mu_8^2 + 2\mu_{12}\mu_4).
\end{aligned}$$

**Remark 3.18.** We denote by  $\mathcal{C}^\circ$  the *regular polygon* obtained by cutting off  $\mathcal{C}$  by a set of  $g$  paths passing through  $\infty$  which represents a symplectic base of  $H_1(\mathcal{C}, \mathbb{Z})$ . Let  $\omega$  and  $\eta$  are any two 1-forms of the second kind with no poles elsewhere  $\infty$ , and we regard them as elements in  $H^1(\mathcal{C}, \mathbb{C})$  via (3.1). As usual, the space  $H^1(\mathcal{C}, \mathbb{C})$  is equipped with the inner product defined by

$$\omega \star \eta = \int_{\partial \mathcal{C}^\circ} \left( \int_{\infty}^P \omega(P) \right) \eta(P) = \sum_{\mathcal{C}^\circ} \text{Res} \left( \int_0^t \omega(t) \right) \eta(t).$$

Then the set of  $\{\omega_{w_j}\}$  and  $\{\eta_{-w_j}\}$  gives a symplectic base of the space  $H^1(\mathcal{C}, \mathbb{C})$  (see [12]). Indeed, if  $\omega, \eta \in \{\omega_{w_j}, \eta_{-w_j} \mid j = 1, \dots, g\}$ , we have

$$\omega \star \eta = \text{Res}_{t=0} \left( \int_{\text{formal}}^t \omega(t) \right) \eta(t)$$

and symplectic property follows from the expansion we obtained before.

**Example 3.19.** Here, we shall show the integrals for the case  $(e, q) = (3, 4)$ :

$$\begin{aligned}
\left( \int_{\text{formal}}^t \omega_5(t) \right) \eta_{-5}(t) &= (t^{-1} - \frac{8}{35}\mu_2t + \dots)dt, \quad \left( \int_{\text{formal}}^t \omega_5(t) \right) \eta_{-2}(t) = (\frac{2}{5}t^2 - \frac{4}{7}\mu_2t^4 + \dots)dt, \\
\left( \int_{\text{formal}}^t \omega_5(t) \right) \eta_{-1}(t) &= (\frac{1}{5}t^3 - \frac{2}{7}\mu_2t^5 + \dots)dt, \\
\left( \int_{\text{formal}}^t \omega_2(t) \right) \eta_{-5}(t) &= (\frac{5}{2}t^{-4} + \frac{7}{4}\mu_2t^{-2} - \frac{1}{6}\mu_2^2 + \dots)dt, \\
\left( \int_{\text{formal}}^t \omega_2(t) \right) \eta_{-2}(t) &= (t^{-1} - \frac{1}{2}\mu_2t + \dots)dt, \quad \left( \int_{\text{formal}}^t \omega_2(t) \right) \eta_{-1}(t) = (\frac{1}{2} - \frac{1}{4}\mu_2t^2 + \dots)dt, \\
\left( \int_{\text{formal}}^t \omega_1(t) \right) \eta_{-5}(t) &= (5t^{-5} + \frac{13}{3}\mu_2t^{-3} - \frac{3}{4}\mu_3t^{-2} - (-\frac{1}{2}\mu_3\mu_2 + \frac{5}{3}\mu_5) + \dots)dt, \\
\left( \int_{\text{formal}}^t \omega_1(t) \right) \eta_{-2}(t) &= (2t^{-2} - \frac{2}{3}\mu_2 + \dots)dt, \quad \left( \int_{\text{formal}}^t \omega_1(t) \right) \eta_{-1}(t) = (t^{-1} - \frac{1}{3}\mu_2t + \dots)dt.
\end{aligned}$$



## 4 The standard theta cycles

We fix here some notations including *standard theta cocycles*. Firstly we define the discriminant of  $\mathcal{C}$ .

**Definition 4.1.** Assuming all the  $\mu_j$ s are indeterminates, we define

$$\begin{aligned} R_1 &= \text{rslt}_x \left( \text{rslt}_y \left( f(x, y), \frac{\partial}{\partial x} f(x, y) \right), \text{rslt}_y \left( f(x, y), \frac{\partial}{\partial y} f(x, y) \right) \right), \\ R_2 &= \text{rslt}_y \left( \text{rslt}_x \left( f(x, y), \frac{\partial}{\partial x} f(x, y) \right), \text{rslt}_x \left( f(x, y), \frac{\partial}{\partial y} f(x, y) \right) \right), \\ R &= \gcd(R_1, R_2) \quad \text{in } \mathbb{Z}[\boldsymbol{\mu}]. \end{aligned}$$

Here  $\text{rslt}_z$  stands for taking the resultant of Sylvester. Then  $R$  is undoubtedly a square element in the ring  $\mathbb{Z}[\boldsymbol{\mu}]$ . However, the author does not have any proof of this. We can check this by computer for the cases  $(d, q) = (2, 3), (2, 5), (3, 4)$ . No proof for this is not a defect for our context. We denote by  $D$  one of the square root of  $R$ . We take the signature of this square root with adding (5.3) as the equality 6.5 exactly holds. After all these operation, in order to get the correct *discriminant*  $D$ , we substitute the own value of  $\mu_j$  to  $D$ . The reason for assuming  $\{\mu_j\}$  be indeterminates is because taking greatest common divisor and substitution for  $\{\mu_j\}$  some values are not commutative.

If all  $\mu_j$ s belong to  $\mathbb{C}$  and the discriminant  $D$  of  $\mathcal{C}$  is not 0, then

$$\Lambda = \left\{ \oint (\omega_{w_1}, \dots, \omega_{w_g}) \text{ for any closed paths on } \mathcal{C} \right\} \subset \mathbb{C}^g$$

is a lattice of the space  $\mathbb{C}^g$ . Then for each  $k \geq 1$ , we define *Abel-Jacobi map* by

$$\begin{aligned} \iota : \text{Sym}^k \mathcal{C} &\longrightarrow J, \\ (P_1, \dots, P_k) &\mapsto \sum_{j=1}^k \int_{\infty}^{P_j} (\omega_{w_1}, \dots, \omega_{w_g}) \mod \Lambda. \end{aligned}$$

Its image is denoted by  $W^{[k]}$ . We assume  $W^{[0]}$  is a one point set consists of the origin of  $J$ . Moreover, we define

$$(4.2) \quad \Theta^{[k]} = W^{[k]} \cup [-1]W^{[k]}, \quad (k \geq 0),$$

where  $[-1]$  is an involution which transform  $u_{w_1}, \dots, u_{w_g}$  to  $-u_{w_1}, \dots, -u_{w_g}$ . We know that  $W^{[k]} = \Theta^{[k]}$  for  $k \geq g-1$ , and  $W^{[k]} = \Theta^{[k]} = J$  for  $k \geq g$ . We call  $\Theta^{[k]}$  ( $0 \leq k \leq g-1$ ) the *standard theta cocycles* and call  $\Theta^{[g-1]}$  the *standard theta divisor* of  $J$ .

## 5 Definition of the sigma function

In this section, we define the sigma function precisely. For proofs, the reader is referred to [14].

We have already defined the 2-forms  $\{\eta_{w_j}\}$  of second kind. Let introduce four  $g \times g$  matrices

$$\begin{aligned}\omega' &= \left[ \int_{\alpha_j} \omega_{w_{g-i+1}} \right], & \omega'' &= \left[ \int_{\beta_j} \omega_{w_{g-i+1}} \right], \\ \eta' &= \left[ \int_{\alpha_j} \eta_{w_{g-i+1}} \right], & \eta'' &= \left[ \int_{\beta_j} \eta_{w_{g-i+1}} \right].\end{aligned}$$

For an arbitrary  $u \in \mathbb{C}^g$ , we define  $u'$  and  $u'' \in \mathbb{R}^g$  by

$$(5.1) \quad u = \omega' u' + \omega'' u''.$$

We write as  $\omega'^{-1t}(\omega_{w_g}, \dots, \omega_{w_1}) = {}^t(\hat{\omega}_1, \dots, \hat{\omega}_g)$ ,  $\omega'^{-1}\omega'' = [T_{ij}]$ , and define

$$\begin{aligned}\delta_j &= -\frac{1}{2}T_{jj} - \int_{\infty}^{I_j} \hat{\omega}_j + \sum_{i=1}^g \int_{\alpha_i} \left( \int_{\infty}^P \hat{\omega}_j \right) \hat{\omega}_i(P), \\ \delta &= \omega' {}^t(\delta_1, \dots, \delta_g).\end{aligned}$$

Here, the integral are given on the regular polygon  $\mathcal{C}_0$  (defined in 3.18) whose boundary is  $\alpha_1 \circ \alpha_1^{-1} \circ \beta_1 \circ \beta_1^{-1} \circ \dots \circ \alpha_g \circ \alpha_g^{-1} \circ \beta_g \circ \beta_g^{-1}$  after regarding  $\mathcal{C}$  as a compact Riemann surface, and  $I_j$  is the initial point of  $\alpha_j$  corresponding  $\infty$ . The vector  $\delta$  is the *Riemann constant* determined by  $\mathcal{C}$  with our base point  $\infty$ . Since the divisor of  $\omega_{w_g}$  is  $(2g-2)\infty$ , we have

$$\delta \in \frac{1}{2}\Lambda.$$

Following the definition (5.1), we define  $\delta'$  and  $\delta'' \in \frac{1}{2}\mathbb{Z}^g$  by  $\delta = \omega' \delta' + \omega'' \delta''$ . For any  $u, v \in \mathbb{C}^g$  and any  $\ell \in \Lambda$ , we define

$$\begin{aligned}L(u, v) &= {}^t u (v' \eta' + v'' \eta''), \\ \chi(\ell) &= \exp \left( 2\pi i (\ell' {}^t \delta' + \ell'' {}^t \delta'' + \frac{1}{2} \ell' {}^t \ell'') \right).\end{aligned}$$

Then a theorem of Frobenius shows that the entire functions  $\sigma(u)$  on the space  $\mathbb{C}^g$  satisfying the *translational relation*

$$(5.2) \quad \sigma(u + \ell) = \chi(\ell) \sigma(u) \exp L(u + \frac{1}{2}\ell, \ell), \quad (u \in \mathbb{C}^g, \ell \in \Lambda)$$

form a 1-dimensional vector space. Such functions has zeroes of order 1 exactly along the pull-back image of  $\Theta^{[g-1]}$  via the canonical map  $\kappa : \mathbb{C}^g \rightarrow \mathbb{C}^g/\Lambda = J$ .

We shall describe this in other words. There exists a vector of functions

$$\zeta(u) = [\zeta_{w_1}(u), \dots, \zeta_{w_g}(u)]$$

having properties

$$\zeta(u + \ell) = \zeta(u) + \eta' \ell' + \eta'' \ell'' \quad (\ell \in \Lambda)$$

and that there exists unique entire function  $\sigma(u)$  on  $\mathbb{C}^g$ , up to multiplication of a constant, which has zeroes of order 1 exactly along  $\kappa^{-1}(\Theta^{[g-1]})$  and satisfies

$$d \log \sigma(u) = \zeta_{w_1}(u) du_{w_1} + \zeta_{w_2}(u) du_{w_2} + \dots + \zeta_{w_g}(u) du_{w_g} (= \zeta(u) du).$$

## Construction of the sigma function via a theta series

We define a function  $\sigma(u)$  by

$$\sigma(u) = c \exp \left( -\frac{1}{2} {}^t u \eta' \omega'^{-1} u \right) \vartheta \left[ \begin{smallmatrix} \delta'' \\ \delta' \end{smallmatrix} \right] (\omega'^{-1} u; \omega'^{-1} \omega''),$$

and call it the *sigma function* for  $\mathcal{C}$ . Here,  $c \neq 0$  is a constant independent of  $u$  is not important in this paper. We define  $c$  as a constant by which the expression 6.5 holds. The correct value of  $c$  might given by

$$(5.3) \quad c = \frac{1}{D^{1/8}} \left( \frac{|\omega'|}{(2\pi)^g} \right)^{1/2},$$

where  $\pi = 3.1415 \dots$  is the circle ratio,  $|\omega'|$  is the determinant of the matrix  $\omega'$ , and  $D$  is the discriminant defined by 4.1. However, the choice of the signature of the right hand side in (5.3) is not clear for the author.

If we choose another symplectic base of  $H_1(\mathcal{C}, \mathbb{Z})$  the period integral  $\omega'$ ,  $\omega''$ ,  $\eta'$ ,  $\eta''$  and the Riemann constants  $\delta'$ ,  $\delta''$  are transformed. But  $\sigma(u)$  itself is invariant under this change, which property is proved by using modular transformation property of the theta series (see [1], p.536). Of course, the translational formula for the theta series with respect to the change  $u$  by  $u + \ell$  implies (5.2).

## 6 The determinantal expression of the sigma function

Now we proceed to explain the determinantal expression of the sigma function. We use the arithmetic parameter  $t$  in place of the local parameter  $z$  used in [12]. Moreover we denote the set of infinitely many number of variables  $\{t_j\}$  used in [12] by  $\{U_j\}$ , which is used in the theory of tau functions in integrable systems. We write as<sup>1</sup>

$$(6.1) \quad \log \left( \sqrt{\frac{1}{t^{2g-2}} \frac{\omega_{w_g}(t)}{dt}} \right) = \sum_j \frac{c_j}{j} t^j.$$

$$(6.2) \quad \omega_{w_g}(t) \in t^{w_g} \mathbb{Z}[\boldsymbol{\mu}][[t]] dt$$

So that,

$$c_j \in \frac{1}{2} \mathbb{Z}[\boldsymbol{\mu}].$$

It is important for our main theorem that, for which  $j$ , we have  $c_j \in \mathbb{Z}[\boldsymbol{\mu}]$ .

We define  $q_{ij} \in \mathbb{Z}[\boldsymbol{\mu}]$  by

$$\boldsymbol{\xi}(t_1, t_2) - \frac{dt_1 dt_2}{(t_1 - t_2)^2} = \sum_{i,j} q_{ij} t_1^{i-1} t_2^{j-1} dt_1 dt_2.$$

---

<sup>1</sup>Please do not confuse with  $c_j$  in 3.16.

Using  $\{q_{ij}\}$ , we set

$$q(u) = \sum_{j=1}^g c_{w_j} U_{w_j} + \frac{1}{2} \sum_{i=1}^g \sum_{j=1}^g q_{ij} U_{w_i} U_{w_j}.$$

Since  $\frac{1}{2}U_{w_i}^2 = \frac{1}{2!}U_{w_i}^2$  and  $q_{ij} = q_{ji} \in \mathbb{Z}[\boldsymbol{\mu}]$ , the later sum is Hurwitz integral with respect to  $\{U_j\}$ . For a monomial  $\varphi(t)$  of  $x(t)$  and  $y(t)$ , we write its coefficient as

$$\varphi(t) = \sum_j (\varphi)_{(j)} t^j.$$

We display all the monomials of  $x(t)$  and  $y(t)$  in increasing order of the order of the pole at  $t = 0$ . We define a matrix  $\Gamma$  with numbering by integers in lows and doing by non-negative integers by

$$(6.3) \quad \Gamma = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \varphi_{v_3}(t)_{(-6)} & \varphi_{v_2}(t)_{(-6)} & \varphi_{v_1}(t)_{(-6)} & \varphi_0(t)_{(-6)} \\ \cdots & \varphi_{v_3}(t)_{(-5)} & \varphi_{v_2}(t)_{(-5)} & \varphi_{v_1}(t)_{(-5)} & \varphi_0(t)_{(-5)} \\ \cdots & \varphi_{v_3}(t)_{(-4)} & \varphi_{v_2}(t)_{(-4)} & \varphi_{v_1}(t)_{(-4)} & \varphi_0(t)_{(-4)} \\ \cdots & \varphi_{v_3}(t)_{(-3)} & \varphi_{v_2}(t)_{(-3)} & \varphi_{v_1}(t)_{(-3)} & \varphi_0(t)_{(-3)} \\ \hline \cdots & \varphi_{v_3}(t)_{(-2)} & \varphi_{v_2}(t)_{(-2)} & \varphi_{v_1}(t)_{(-2)} & \varphi_0(t)_{(-2)} \\ \cdots & \varphi_{v_3}(t)_{(-1)} & \varphi_{v_2}(t)_{(-1)} & \varphi_{v_1}(t)_{(-1)} & \varphi_0(t)_{(-1)} \\ \cdots & \varphi_{v_3}(t)_{(0)} & \varphi_{v_2}(t)_{(0)} & \varphi_{v_1}(t)_{(0)} & \varphi_0(t)_{(0)} \\ \cdots & \varphi_{v_3}(t)_{(1)} & \varphi_{v_2}(t)_{(1)} & \varphi_{v_1}(t)_{(1)} & \varphi_0(t)_{(1)} \\ \cdots & \varphi_{v_3}(t)_{(2)} & \varphi_{v_2}(t)_{(2)} & \varphi_{v_1}(t)_{(2)} & \varphi_0(t)_{(2)} \\ \ddots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Let  $T$  be a new indeterminate, set

$$\mathbf{T} = U_1 T + U_2 T^2 + U_3 T^3 + U_4 T^4 + U_5 T^5 + \cdots.$$

(For the  $(3, 4)$ -curve, this will be set  $= U_1 T + U_2 T^2 + U_5 T^5$  later.)

We define  $p_j \in \mathbb{Q}[\boldsymbol{\mu}]\langle\langle U \rangle\rangle$  by

$$\begin{aligned} \mathbf{p} &= 1 + \frac{1}{1!}\mathbf{T} + \frac{1}{2!}\mathbf{T}^2 + \frac{1}{3!}\mathbf{T}^3 + \frac{1}{4!}\mathbf{T}^4 + \frac{1}{5!}\mathbf{T}^5 + \cdots \\ &= p_0 + p_1 T + p_2 T^2 + p_3 T^3 + p_4 T^4 + p_5 T^5 + \cdots. \end{aligned}$$

Then, we see

$$p_j \in \mathbb{Z}[\boldsymbol{\mu}]\langle\langle U \rangle\rangle \cap \mathbb{Q}[\boldsymbol{\mu}][U].$$

Using those matrices, we define

$$(6.4) \quad S(U) = S(B^{-1}u) = \left[ \begin{array}{cccccc|cccccc} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & 1 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & \cdots \\ \cdots & 0 & 1 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & \cdots \\ \cdots & 0 & 0 & 1 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & p_1 & p_2 & p_3 & p_4 & p_5 & \cdots \end{array} \right].$$

We explain the main result of [12]. Let

$$B = \begin{bmatrix} \omega_{w_g}(t)_{(0)} & \omega_{w_g}(t)_{(1)} & \omega_{w_g}(t)_{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \omega_{w_2}(t)_{(0)} & \omega_{w_2}(t)_{(1)} & \omega_{w_2}(t)_{(2)} & \cdots \\ \omega_{w_1}(t)_{(0)} & \omega_{w_1}(t)_{(1)} & \omega_{w_1}(t)_{(2)} & \cdots \end{bmatrix}.$$

We need the following

**Theorem 6.5.** (Nakayashiki [12]) *We have*

$$\sigma(BU) = \det(S(U)\Gamma) \cdot \exp q(U).$$

Here, the product of matrices is operated as the lines in (6.3) and in (6.4) engage. We define (the sign of) the overall multiplicative constant  $c$  of  $\sigma(u)$  by this equality.

**Example 6.6.** In the case  $(e, q) = (3, 4)$ , if all the  $\mu_j = 0$ , then  $\sigma(u)$  is given by

$$\sigma(u_5, u_3, u_1)|_{\mu=0} = \begin{vmatrix} p_3 & p_4 & p_5 \\ 1 & p_1 & p_2 \\ & 1 & p_1 \end{vmatrix} = u_5 - u_1 u_2^2 + \frac{1}{20} u_1^5.$$

We explain the weight of  $\sigma(u)$ .

**Proposition 6.7.** *The expansion of  $\sigma(u)$  around the origin is of homogeneous weight. Its weight is*

$$\text{wt}(\sigma(u)) = \frac{1}{24}(e^2 - 1)(q^2 - 1).$$

*Proof.* (Due to [5], p.97) For our Weierstrass sequence (2.5)  $(w_1, w_2, \dots)$  is the ascending set of positive integers that are not representable in the form  $ae + bq$  with non-negative integers  $a$  and  $b$ . Set  $w(T) = \sum_i T^{w_i}$  with an indeterminate  $T$ . We have

$$w(T) = \frac{1}{1-T} - \frac{1-T^{eq}}{(1-T^e)(1-T^q)}.$$

Therefore the length of the Weierstrass sequence  $g = w(1)$ , and the sum  $G = \frac{d}{dT}w(1)$  of the elements of this sequence are given by the formulae

$$g = \frac{(e-1)(q-1)}{2}, \quad G = \frac{eq(e-1)(q-1)}{4} - \frac{(e^2-1)(q^2-1)}{12}.$$

Let us introduce new variables  $\{u^{(1)}, \dots, u^{(g)}\}$ , with weight 1 for all, such that the relations  $u_j = \frac{1}{j} \sum_{i=1}^g u^{(i)j}$  holds for  $j = w_1, \dots, w_g$  (Newton's symmetric polynomial). Over any algebraically closed field, by [8] p.29,  $\ell. - 4$  and p.28,  $\ell.13$  for example, we see that for any value of  $(u_{w_g}, \dots, u_{w_1})$  there exists a solution of the system of that relations, and we know that

$$\sigma(u_{w_g}, u_{w_{g-1}}, \dots, u_{w_1})|_{\mu=0} = \frac{\det[u^{(i)w_j}]}{\det[u^{(i)j-1}]} \quad (i, j = 1, \dots, g).$$

Hence,

$$\text{wt}(\sigma(u_{w_g}, \dots, u_{w_1})) = \sum_{k=1}^g w_k - \sum_{k=1}^g (k-1) = G - \frac{(g-1)g}{2} = \frac{(e^2-1)(q^2-1)}{24}.$$

□

Using Theorem 6.5, we prove that the expansion  $\sigma(u)$  around the origin is of Hurwitz integral. We let denote by  $B_0$  the matrix obtained by removing the columns except for  $j = 1, \dots, g$  from  $B$ . The matrix  $B_0$  is an upper triangular matrix belongs to  $\text{SL}(g, \mathbb{Z}[\mu])$  with all its diagonal entries being all 1. We set all  $U_i = 0$  except for  $U_{w_j}$  ( $j = 1, \dots, g$ ), and set, for the variables  $\{u_{w_j}\}$  of  $\sigma(u)$ ,

$$(6.8) \quad [U_{w_g} \ \cdots \ U_{w_2} \ U_{w_1}] = B_0^{-1} [u_{w_g} \ \cdots \ u_{w_2} \ u_{w_1}] = B_0^{-1} u.$$

Then  $U_{w_j} \in \mathbb{Z}[\mu][u]$  and the following expression of the sigma function by Theorem 6.5.

**Theorem 6.9.** *The sigma function is expressed as*

$$(6.10) \quad \sigma(u) = \det(S(B_0^{-1}u)\Gamma) \cdot \exp q(B_0^{-1}u).$$

This is the main tool to prove our main theorem. Since  $p_j$  is Hurwitz integral as a polynomial of  $\{u_{w_i}\}$ , we see  $\det(S(B_0^{-1}u)\Gamma)$  is Hurwitz integral. The crucial part of the proof is on the expansion of  $\exp q(B_0^{-1}u)$ .

**Example 6.11.** For the  $(3, 4)$ -curve, the set  $\{\varphi_{v_i}\}$  consists of

$$\cdots, x^3, y^2, xy, x^2, y, x, 1.$$

Taking their expansion with respect to  $t$ , we define

$$\Gamma = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & (x^3)_{(-4)} & (y^2)_{(-4)} & (xy)_{(-4)} & (x^2)_{(-4)} & (y)_{(-4)} & (x)_{(-4)} & (1)_{(-4)} \\ \cdots & (x^3)_{(-3)} & (y^2)_{(-3)} & (xy)_{(-3)} & (x^2)_{(-3)} & (y)_{(-3)} & (x)_{(-3)} & (1)_{(-3)} \\ \cdots & (x^3)_{(-2)} & (y^2)_{(-2)} & (xy)_{(-2)} & (x^2)_{(-2)} & (y)_{(-2)} & (x)_{(-2)} & (1)_{(-2)} \\ \cdots & (x^3)_{(-1)} & (y^2)_{(-1)} & (xy)_{(-1)} & (x^2)_{(-1)} & (y)_{(-1)} & (x)_{(-1)} & (1)_{(-1)} \\ \cdots & (x^3)_{(0)} & (y^2)_{(0)} & (xy)_{(0)} & (x^2)_{(0)} & (y)_{(0)} & (x)_{(0)} & (1)_{(0)} \\ \cdots & (x^3)_{(1)} & (y^2)_{(1)} & (xy)_{(1)} & (x^2)_{(1)} & (y)_{(1)} & (x)_{(1)} & (1)_{(1)} \\ \cdots & (x^3)_{(2)} & (y^2)_{(2)} & (xy)_{(2)} & (x^2)_{(2)} & (y)_{(2)} & (x)_{(2)} & (1)_{(2)} \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

As we explained before, let define  $c_j$ s by

$$\begin{aligned} \log \sqrt{\frac{1}{t^{6-2}} \frac{\omega_5(t)}{dt}} &= \sum_j \frac{c_j}{j} t^j \\ &= -\mu_1 t + (\mu_1^2 - 2\mu_2) \frac{1}{2} t^2 + (-\mu_1^3 + 3\mu_2\mu_1 + 3\mu_3) \frac{1}{3} t^3 \\ &\quad + (\mu_1^4 - 4\mu_2\mu_1^2 - 4\mu_3\mu_1 - 6\mu_4 + 2\mu_2^2) \frac{1}{4} t^4 \\ &\quad + (-\mu_1^5 + 5\mu_2\mu_1^3 + 5\mu_3\mu_1^2 + 5(3\mu_4 - \mu_2^2)\mu_1 - 5\mu_3\mu_2 - \frac{15}{2}\mu_5) \frac{1}{5} t^5 \\ &\quad + \cdots. \end{aligned}$$

Then

$$(6.12) \quad \begin{cases} c_1 = -\mu_1, \\ c_2 = \mu_1^2 - 2\mu_2, \\ c_5 = -\mu_1^5 + 5\mu_2\mu_1^3 + 5\mu_3\mu_1^2 + 5(3\mu_4 - \mu_2^2)\mu_1 - 5\mu_3\mu_2 - \boxed{\frac{15}{2}\mu_5}. \end{cases}$$

The boxed last term gives arise some denominator. The elements  $q_j \in \mathbb{Z}[\boldsymbol{\mu}]$  are defined by

$$\begin{aligned} \xi(t_1, t_2) &= \frac{dt_1 dt_2}{(t_1 - t_2)^2} \\ &= \sum_{i,j} q_{ij} t_1^{i-1} t_2^{j-1} \\ &= \mu_4 t_1^2 + 2\mu_4 t_2 t_1 + \mu_4 t_2^2 \\ &\quad + (\mu_5 - 2\mu_4 \mu_1) t_1^3 + (2\mu_5 - 4\mu_4 \mu_1) t_2 t_1^2 + (2\mu_5 - 4\mu_4 \mu_1) t_2^2 t_1 + (\mu_5 - 2\mu_4 \mu_1) t_2^3 \\ &\quad + (3\mu_4 \mu_1^2 - 2\mu_5 \mu_1 - 2\mu_2 \mu_4 - \mu_6) t_1^4 + (6\mu_4 \mu_1^2 - 4\mu_5 \mu_1 - 4\mu_2 \mu_4 - 2\mu_6) t_2^3 t_1 \\ &\quad + (7\mu_4 \mu_1^2 - 5\mu_5 \mu_1 - 4\mu_2 \mu_4 - 3\mu_6) t_2^2 t_1^2 + (6\mu_4 \mu_1^2 - 4\mu_5 \mu_1 - 4\mu_2 \mu_4 - 2\mu_6) t_2 t_1^3 \\ &\quad + \dots, \end{aligned}$$

and the necessary elements among them are given explicitly by

$$\left\{ \begin{array}{l} q_{11} = 0, \\ q_{21} = 0, \\ q_{12} = 0, \\ q_{22} = -2\mu_4, \\ q_{51} = -3\mu_4 \mu_1^2 + 2\mu_5 \mu_1 + (2\mu_2 \mu_4 + \mu_6), \\ q_{15} = -3\mu_4 \mu_1^2 + 2\mu_5 \mu_1 + (2\mu_2 \mu_4 + \mu_6), \\ q_{52} = 8\mu_4 \mu_1^3 - 6\mu_5 \mu_1^2 + (-12\mu_2 \mu_4 - 4\mu_6) \mu_1 + (-4\mu_3 \mu_4 + 4\mu_5 \mu_2), \\ q_{25} = 8\mu_4 \mu_1^3 - 6\mu_5 \mu_1^2 + (-12\mu_2 \mu_4 - 4\mu_6) \mu_1 + (-4\mu_3 \mu_4 + 4\mu_5 \mu_2), \\ q_{55} = -23\mu_4 \mu_1^6 + 21\mu_5 \mu_1^5 + (96\mu_2 \mu_4 + 19\mu_6) \mu_1^4 \\ \quad + (60\mu_3 \mu_4 - 68\mu_5 \mu_2) \mu_1^3 + (56\mu_4^2 - 94\mu_2^2 \mu_4 - 44\mu_6 \mu_2 - 38\mu_3 \mu_5 - 34\mu_8) \mu_1^2 \\ \quad + ((-72\mu_3 \mu_2 - 44\mu_5) \mu_4 + 40\mu_5 \mu_2^2 - 20\mu_6 \mu_3 - 17\mu_9) \mu_1 \\ \quad - 24\mu_2 \mu_4^2 + (12\mu_2^3 - 9\mu_3^2 - 12\mu_6) \mu_4 + (11\mu_6 \mu_2^2 + 19\mu_3 \mu_5 + 18\mu_8) \mu_2 + 5\mu_5^2. \end{array} \right.$$

Setting all  $U_j$  to be 0 except  $U_1$ ,  $U_2$ , and  $U_5$ . Then the relation with the variables of  $\sigma(u) = \sigma(u_5, u_2, u_1)$  is given by

$$\begin{aligned} U_1 &= u_1 + \mu_1 u_2 + (\mu_2 \mu_1^2 + \mu_3 \mu_1 + 2\mu_4 - \mu_2^2) u_5, \\ U_2 &= u_2 + (\mu_1^3 - 2\mu_2 \mu_1 - \mu_3) u_5, \\ U_5 &= u_5. \end{aligned}$$

Using those elements, we have

$$\begin{aligned}
q(U) &= q(B_0^{-1} [u_5, u_2, u_1]) \\
&= c_1 U_1 + c_2 U_2 + c_5 U_5 \\
&\quad - \frac{1}{2}(q_{11} U_1^2 + (q_{12} + q_{21}) U_2 U_1 + (q_{15} + q_{51}) U_1 U_5 \\
&\quad + q_{22} U_2^2 + (q_{52} + q_{25}) U_5 U_2 + q_{55} U_5^2).
\end{aligned}$$

In our case,

$$\begin{aligned}
p_1 &= u_1 + \mu_1 u_2 + (\mu_2 \mu_1^2 + \mu_3 \mu_1 + 2\mu_4 - \mu_2^2) u_5, \\
p_2 &= u_2 + \frac{1}{2} u_1^2 + \frac{1}{2} \mu_1^2 u_2^2 + (\mu_1^3 - 2\mu_2 \mu_1 - \mu_3) u_5 \\
&\quad + \mu_1 u_2 u_1 + (\mu_2 \mu_1^2 + \mu_3 \mu_1 + 2\mu_4 - \mu_2^2) u_5 u_1 \\
&\quad + (\mu_2 \mu_1^3 + \mu_3 \mu_1^2 + 2\mu_4 \mu_1 - \mu_2^2 \mu_1) u_5 u_2 \\
&\quad + (\frac{1}{2} \mu_2^2 \mu_1^4 + \mu_3 \mu_2 \mu_1^3 + (2\mu_2 \mu_4 - \mu_2^3 + \frac{1}{2} \mu_3^2) \mu_1^2 \\
&\quad + (2\mu_3 \mu_4 - \mu_3 \mu_2^2) \mu_1 + 2\mu_4^2 - 2\mu_2^2 \mu_4 + \frac{1}{2} \mu_2^4) u_5^2, \\
p_3 &= \frac{1}{6} u_1^3 + u_2 u_1 + \text{“higher weight terms in } u_1, u_2, u_5\text{”}, \\
p_4 &= \frac{1}{24} u_1^4 + \frac{1}{2} u_2 u_1^2 + \frac{1}{2} u_2^2 + \text{“higher weight terms in } u_1, u_2, u_5\text{”}, \\
p_5 &= \frac{1}{120} u_1^5 + \frac{1}{6} u_2 u_1^3 + \frac{1}{2} u_2^2 u_1 + u_5 + \text{“higher weight terms in } u_1, u_2, u_5\text{”},
\end{aligned}$$

and so on.

The rest of this paper is devoted to investigate how does the prime 2 occur in denominators, in general, like the denominator 2 of  $\frac{\mu_5}{2}$  in (6.12).

## 7 The last step of the proof

In this section, we analyze the exponential part of Nakayashi’s expression (6.10), and complete the proof of the main result. Let

$$\tilde{f}(t, s) = s - t^2 - G, \quad G = G(t, s).$$

Concerning (6.1), it is sufficient to check only the odd power term of  $t$  in the expansion of  $\omega_{w_g}(t)$  with respect to  $t$ . Because of the identity

$$\frac{1}{\tilde{f}_s} = \frac{1}{1 - G_s} = (1 + G_s)(1 + G_s^2)(1 + G_s^{2^2})(1 + G_s^{2^3}) \cdots, \quad G_s = \frac{\partial}{\partial s} G,$$

the fact that the polynomial  $G(t, s)$  is a multiple of  $s^2$ , and the general equality  $(X + Y)^{2^k} \equiv X^{2^k} + Y^{2^k} \pmod{2}$ , letting

$$\tilde{f} = s - t^2 - \sum_{j \geq 2} f_j(t) s^j, \quad (f_j(t) \in \mathbb{Z}[\mu][t]),$$

we have

$$G_s = 1 - \tilde{f}_s = \sum_{j \geq 2} j f_j(t) s^{j-1}.$$



**Lemma 7.1.** *Writing down  $\omega_{w_g}(t)$  in terms of  $t$  and  $s$ , it is of type*

$$\frac{t^{\text{even}} s^{\text{odd}}}{\tilde{f}_s} dt \quad \text{or} \quad \frac{t^{\text{even}} s^{\text{even}}}{\tilde{f}_s} dt.$$

*On the other hand the pair of parities of powers of  $x$  and  $y$  in a term of  $f(x, y)$  corresponds to the pair of parities of powers of  $t$  and  $s$  in a term of  $\tilde{f}(t, s)$ . More precisely, a term of the type  $x^{\text{odd}} y^{\text{odd}}$  corresponds the term of the type  $t^{\text{odd}} s^{\text{even}}$  in the former case, and does the terms of the type  $t^{\text{odd}} s^{\text{odd}}$  in the latter case.*

*Proof.* Of course, in any case, the parity of  $a$  and  $b$  is different. If  $e$  is even, then both  $q$  and  $b$  are odd. In this case,  $c$  is odd and  $d$  is even. If  $e$  is odd, then  $d$  is odd. Summarizing the deduction, in each case separated by the parities of  $e, a, b, c, d$ , it is easy to check the claim by using (2.3) and (2.7).  $\square$

We note here that, in

$$\log \sqrt{\frac{\omega_{w_g}(t)}{t^{2g-2} dt}} = \sum_{j=1}^{\infty} \frac{c_j}{j} t^{j-1},$$

namely, in the expression

$$\sum_{j=1}^{\infty} c_j t^{j-1} = \frac{1}{2} \frac{d}{dt} \log \frac{\omega_{w_g}(t)}{t^{2g-2} dt} = \frac{\frac{1}{2} \frac{d}{dt} (\omega_{2g-2}(t)/dt)}{\omega_{2g-2}(t)/dt} = \frac{\frac{1}{2} \frac{d}{dt} (1 + \dots)}{1 + \dots},$$

all the coefficients in the denominators belongs to  $\mathbb{Z}[\boldsymbol{\mu}]$ . In the coefficients in the series inside  $\frac{d}{dt}$ , the coefficients of the term of type  $t^{\text{even}}$  is canceled out by the denominator 2 of  $\frac{1}{2}$ . Therefore, the critical case occurs only concerning the coefficients of odd power terms in the expansion of  $\omega_{w_g}(t)$ .

In the following, we check the two cases in Lemma above separately.

*The former case.* In

$$\frac{s^{\text{odd}} t^{\text{even}}}{\tilde{f}_s} = t^{\text{even}} s^{\text{even}} \frac{s}{1 - G_s} = t^{\text{even}} s^{\text{even}} s(1 + G_s)(1 + G_s^2)(1 + G_s^{2^2})(1 + G_s^{2^3}) \dots,$$

it is sufficient to check  $s(1 + G_s)$ . As an element in  $\mathbb{Z}[\boldsymbol{\mu}][[t]]$ , we have

$$\begin{aligned} s(1 + G_s) &= s + sG_s \\ &\equiv t^2 + G + sG_s \pmod{2} \\ &\equiv t^2 + G + \text{“the sum of the terms of type } s^{\text{odd}} \text{ in } G\text{”} \pmod{2} \\ &\equiv \text{“the sum of the terms of type } s^{\text{even}} \text{ in } \tilde{f}\text{”} \pmod{2}. \end{aligned}$$

In this situation, by replacing the coefficient  $\mu_j$  by  $\mu'_j/2$  in the set  $\boldsymbol{\mu}$  if and only if the  $\mu_j$  occurs as the coefficient of any term of type  $t^{\text{odd}} s^{\text{even}}$  in  $\tilde{f}$  (this corresponds a term of type  $x^{\text{odd}} y^{\text{odd}}$ ), we have the replaced set  $\boldsymbol{\mu}'$ . Then the coefficient of any term of type  $t^{\text{odd}}$  in  $s(1 + G_s) \in \mathbb{Z}[\boldsymbol{\mu}][[t]]$  belongs to  $2\mathbb{Z}[\boldsymbol{\mu}']$ . As a result, we see that all the coefficients of the expansion of

$$\frac{1}{2} \frac{d}{dt} \log \left( \frac{\omega_{w_g}(t)}{t^{2g-2} dt} \right)$$

with respect to  $t$  belong to  $\mathbb{Z}[\boldsymbol{\mu}']$ . Hence,  $\forall c_j \in \mathbb{Z}[\boldsymbol{\mu}']$ . Now our claim is proved for the first case.

*The latter case.* Since

$$\frac{s^{\text{even}} t^{\text{even}}}{\tilde{f}_s} = t^{\text{even}} s^{\text{even}} \frac{1}{1 - G_s} = t^{\text{even}} s^{\text{even}} (1 + G_s)(1 + G_s^2)(1 + G_s^{2^2})(1 + G_s^{2^3}) \cdots .$$

It is sufficient to check  $(1 + G_s)$ . The crucial case may occur for the coefficient of any term of the type  $t^{\text{odd}} s^{\text{odd}}$  in the expansion of  $\tilde{f}$ . Our claim is proved by the same argument in the former case, and we have completed the proof.

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